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A Chern–Simons gauge-fixed Lagrangian in a ‘non-canonical’ BRST approach

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Abstract

This paper presents a possible path which starts from the extended BRST Hamiltonian formalism and ends with a covariant Lagrangian action, using the equivalence between the two formalisms. The approach allows a simple account of the form of the master equation and offers a natural identification of some ‘non-canonical’ operators and variables. These are the main items which solve the major difficulty of the extended BRST Lagrangian formalism, i.e., the gauge-fixing problem. The algorithm we propose applies to a non-Abelian Chern–Simons model coupled with Dirac fields.

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1. Introduction

In the attempt at achieving a coherent quantum description of the gauge-field theories, the BRST technique provides very useful tools. It involves replacing local symmetries with a global one, expressed by a differential operator s which leaves invariant an extended action of the theory. This is the main idea of what is known as the standard BRST symmetry and it has been defined both in the Hamiltonian [1–3] and in the Lagrangian [4–6] approaches. In both cases supplementary variables, called ghosts, are introduced. An extended space of generators and an extended action $S = S_0 + \cdots$ become the main objects of the theory. The nilpotency condition for s leads to the *master equation* for S [6]:

$$s^2 = 0 \Rightarrow (S, S) = 0. \qquad (1)$$

Later on, because of the difficulties the standard BRST approach met in the gauge-fixing procedure which supposes the introduction of some supplementary variables from a *non-minimal sector*, *extended BRST symmetries* have been formulated [7–10]. The main such extension is known as the $sp(2)$ *BRST symmetry* and supposes the existence of two anticommuting differential operators, s_1 and s_2, which can be joined in a symplectic doublet (s_1, s_2) with

$$s = s_1 + s_2; \qquad s^2 = 0. \qquad (2)$$

The $sp(2)$ formalisms, both in the Lagrangian and in the Hamiltonian forms, are presented in [7, 8]. In the Lagrangian case, two antibrackets are defined and the master equations take the form [7]

$$\tfrac{1}{2}(S, S)_a + V_a S = 0; \qquad a = 1, 2. \tag{3}$$

In this paper we will develop an $sp(2)$ approach. We will start from the Hamiltonian formalism and, using its equivalence with the Lagrangian one, we will obtain, in a more simple way and with a better understanding of the significance of the operators $\{V_a, a = 1, 2\}$, the master equations (3). There are two important questions whose answers advocate this approach: why $sp(2)$ and why is it necessary to pass from Hamilton to Lagrange?

The passage from the standard BRST formulation to the $sp(2)$ proved to be highly efficient by solving an important problem: the non-minimal sector, artificially introduced in the standard formulation, appears now in a natural way.

On the other hand, the $sp(2)$ BRST Hamiltonian formalism is easier to be implemented, there are clearly defined rules for choosing an adequate ghost spectrum and there is a simple gauge-fixing procedure [11], non-applicable in the pure Lagrangian approach. Moreover, the use of the $sp(2)$ Hamiltonian formalism offers a great advantage in the quantum approach, especially in the case of open groups [12]. The disadvantage lies in the fact that, in certain cases, it is difficult to put the Hamiltonian formalism in a covariant form. The way to the Lagrangian formalism gives rise to a covariant theory.

In order to be very concrete in our approach, we will use as a working model the non-Abelian Chern–Simons fields coupled with Dirac fields [13]. In recent studies, the terms of the Chern–Simons type are intensively used in the case of topologically massive gauge theories [14] as well as in the case of 3D-gravity [15, 16].

This paper has the following structure: the preliminary considerations are followed by the canonical analysis of the model as presented in section 2. In section 3, we will develop the $sp(2)$ BRST Hamiltonian formalism for our model, and in section 4 we will implement the Lagrangian formalism on the basis of the equivalence between the two approaches. Some concluding remarks will end this paper.

2. Canonical analysis of the model

We will consider an action which describes the non-Abelian Chern–Simons theory in $(2+1)$ dimensions, coupling the non-Abelian gauge fields $\{A_\mu^m, \mu = 0, 1, 2\}$ with the Dirac fields $\{\psi^m, m = 1, 2, \ldots, d\}$. The action will have the form

$$S_0 = \int \mathrm{d}^3x \left(-\frac{1}{4} F_m^{\mu\nu} F_{\mu\nu}^m + \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} \left(\partial_\mu A_\nu^m A_\rho^m + \frac{1}{3} f_{nr}^m A_\mu^m A_\nu^n A_\rho^r \right) + \mathrm{i}\overline{\psi}_{\alpha m} (\gamma^\mu)_n^m (D_\mu)_r^n \psi^{\alpha r} \right), \tag{4}$$

where

$$F_{\mu\nu}^m = \partial_\mu A_\nu^m - \partial_\nu A_\mu^m + f_{nr}^m A_\mu^n A_\nu^r. \tag{5}$$

The Dirac γ-matrices are $\gamma^0 = \sigma^3, \gamma^1 = \mathrm{i}\sigma^1, \gamma^2 = \mathrm{i}\sigma^2$ where σ^1, σ^2 and σ^3 represent the Pauli matrices. The symbol

$$(D_\mu)_n^m = \delta_n^m \partial_\mu + f_{nr}^m A_\mu^r$$

designates the covariant derivative. Each spinor $\psi(x)$ and its adjoint field $\overline{\psi}$ defined as $\overline{\psi} = \psi^+ \gamma^0$ satisfy the Dirac-type equations.

The canonical momenta are defined by

$$p_m^0 \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_0^m} \approx 0, \tag{6}$$

$$p_m^i \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_i^m} = -F_m^{0i} + \frac{k}{2\pi}\varepsilon^{0ij} A_{jm}, \tag{7}$$

$$p_{\alpha m} \equiv \frac{\partial^R \mathcal{L}}{\partial \dot{\psi}^{\alpha m}} = \mathrm{i}\overline{\psi}_{\alpha n}(\gamma^0)^n_m, \tag{8}$$

$$\overline{p}^{\alpha m} \equiv \frac{\partial^L \mathcal{L}}{\partial \dot{\overline{\psi}}_{\alpha m}} \approx 0. \tag{9}$$

The superscripts R and L from relations (8) and (9) designate *right*, respectively, *left* derivatives. The quantities p_m^μ represent the momenta conjugated with A_μ^m. The momenta $p_{\alpha m}, \overline{p}^{\alpha m}$ are conjugated with the Dirac spinors $\psi^{\alpha m}$, respectively, $\overline{\psi}_{\alpha m}$. The conjugation is defined in terms of the Poisson bracket defined in respect of the whole set of the canonical variables: $\left\{q \equiv \left(A_\mu^m, \psi^{\alpha m}, \overline{\psi}_{\alpha m}\right), p \equiv \left(p_m^\mu, p_{\alpha m}, \overline{p}^{\alpha m}\right)\right\}$. For two functionals, F and G, we will have

$$[F, G] \equiv \frac{\delta^R F}{\delta q}\frac{\delta^L G}{\delta p} - (-1)^{\varepsilon_F \varepsilon_G}\frac{\delta^R G}{\delta q}\frac{\delta^L F}{\delta p}. \tag{10}$$

The primary constraints of the model are

$$\chi_m \equiv p_{\alpha m} - \mathrm{i}\overline{\psi}_{\alpha n}(\gamma^0)^n_m \approx 0, \tag{11}$$

$$\overline{\chi}^m \equiv \overline{p}^{\alpha m} \approx 0 \tag{12}$$

and

$$G_{1m} \equiv p_m^0 \approx 0. \tag{13}$$

Constraints (11) and (12) are second class and they will be eliminated by passing to the Dirac brackets. Constraints (13) are first class ones.

The canonical Hamiltonian has the form

$$H_c = \int \mathrm{d}^2x \mathcal{H}_c = \int \mathrm{d}^2x \left(\frac{1}{2}p_i^m p_i^m + \frac{k}{4\pi}\varepsilon^{0ij} p_i^m A_{jm} + \frac{1}{4}F_m^{ij} F_{ij}^m - \frac{1}{2}\left(\frac{k}{4\pi}\right)^2 A_m^i A_i^m \right.$$
$$\left. - A_0^m\left((D_i)^n_m p_n^i + \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_{jm} + \mathrm{i}\overline{\psi}_{\alpha q}(\gamma^0)^q_n f^n_{rm}\psi^{\alpha r}\right) - \mathrm{i}\overline{\psi}_{\alpha q}(\gamma^j)^q_n (D_j)^n_r \psi^{\alpha r}\right). \tag{14}$$

By imposing the time conservation for the primary constraints (13) we have the secondary constraints

$$G_{2m} \equiv (D_i)^n_m p_n^i + \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_{jm} + \mathrm{i}\overline{\psi}_{\alpha q}(\gamma^0)^q_n f^n_{rm}\psi^{\alpha r} \approx 0. \tag{15}$$

The gauge algebra is given by

$$[H_c, G_{1m}] = G_{2m}, \qquad [H_c, G_{2m}] = 0, \tag{16}$$

$$[G_{1m}, G_{1n}] = 0, \qquad [G_{1m}, G_{2n}] = 0, \qquad [G_{2m}, G_{2n}] = 2 f^r_{mn} G_{2r}. \tag{17}$$

3. The $sp(2)$ BRST Hamiltonian formalism

Following the general lines of the $sp(2)$ BRST Hamiltonian approach, we will obtain the most important ingredients for our model, namely the BRST charges and the extended BRST invariant Hamiltonian. By choosing an adequate form for the fermion functional, the gauge-fixed action will be calculated.

The real fields of the theory are $\{A_\mu^m, \mu = 0, 1, 2\}$, $\{\psi^{\alpha m}, \alpha = 0, 1, 2\}$ and $\{\overline{\psi}_{\alpha m}, \alpha = 0, 1, 2\}$ respectively, with $m = 1, \ldots, d$. The Grassmann parities of these fields are $\varepsilon(A_\mu^m) = 0, \varepsilon(\psi^{\alpha m}) = \varepsilon(\overline{\psi}_{\alpha m}) = 1$. The canonical analysis showed that all constraints are merely bosonic. Hence, it is important to mention that, even in the presence of the Dirac fields, the extended phase space for the $sp(2)$ BRST Hamiltonian construction will have the same structure as those corresponding to a pure bosonic theory.

Following the general rules of the $sp(2)$ BRST approach, for each constraint $G_{\Delta m} \equiv \{G_{1m}, G_{2m}\}$ we will introduce two ghost momenta $P_{\Delta ma} \equiv \{P_{1ma}, P_{2ma}, m = 1, \ldots, d; a = 1, 2\}$ which are fermions $\varepsilon(P_{\Delta ma}) = 1$. We define

$$\delta_a P_{\Delta mb} = \delta_{ab} G_{\Delta m}, \qquad \delta_a(\varepsilon_{abc} P_{\Delta mb}) = 0. \tag{18}$$

(no summation after a and b). The non-trivial co-cycles are destroyed by introducing new ghost momenta, $\pi_{\Delta m}$, with $\varepsilon(\pi_{\Delta m}) = 0$ so that

$$\delta_a \pi_{\Delta m} = \varepsilon_{ab} P_{\Delta mb}. \tag{19}$$

No other non-trivial co-cycles appear. We will introduce condensed notation for the ghost momenta generators

$$P_A \equiv \{P_{\Delta ma}, \pi_{\Delta m}, a = 1, 2\}. \tag{20}$$

In conclusion, for assuring the acyclicity of the Koszul–Tate differentials, $\{\delta_a, a = 1, 2\}$ in the $sp(2)$ description, the following set of the momenta (real and ghost types) is necessary

$$\{p, P_A\} \equiv \left\{p_\mu^m, p_\mu^{(1)m}, p_{\alpha m}, \overline{p}^{\alpha m}, P_{\Delta ma}, \pi_{\Delta m}, m = 1, \ldots, d; a = 1, 2; \Delta = 1, 2\right\}. \tag{21}$$

For assuring the canonical structure of the phase space we will introduce the set of the variables (real and ghost)

$$\{q, Q^A\} \equiv \left\{A_m^\mu, A_m^{(1)\mu}, \psi^{\alpha m}, \overline{\psi}_{\alpha m}, Q^{\Delta ma}, \lambda^{\Delta m}, m = 1, \ldots, d; a = 1, 2; \Delta = 1, 2\right\}. \tag{22}$$

The generalized Poisson brackets with respect to the canonical conjugation are defined as in (10).

The Grassmann parities of the ghosts will coincide with those of the conjugated momenta, i.e.

$$\varepsilon(Q^{\Delta ma}) = \varepsilon(P_{\Delta ma}) = 1, \qquad \varepsilon(\lambda^{\Delta m}) = \varepsilon(\pi_{\Delta m}) = 0.$$

The two major problems in the $sp(2)$ BRST Hamiltonian formalism consist of the determination of the two BRST charges and of the extended Hamiltonian. They are formulated as follows:

$$\begin{cases} [\Omega^a, \Omega^b] = 0, a, b = 1, 2 \\ \text{boundary conditions} \end{cases} \tag{23}$$

and

$$\begin{cases} [H, \Omega^a] = 0, a = 1, 2 \\ H|_{P_A=0} = H_0, \end{cases} \tag{24}$$

respectively. These two problems, (23) and (24), can be solved by using the homological perturbation theory. In our model, the BRST charges, solutions of equation (23), will have the form

$$\Omega^a = \int \mathrm{d}^2x\Bigg(p^0_m Q^{1ma} + \left((D_i)^n_m p^i_n + \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_{jm} + \mathrm{i}\overline{\psi}_{\alpha q}(\gamma^0)^q_n f^n_{rm}\psi^{\alpha r}\right)Q^{2ma}$$
$$+ \varepsilon_{ab}(P_{1mb}\lambda^{1m} + P_{2mb}\lambda^{2m}) + f^r_{mn}P_{2rc}Q^{2mc}Q^{2na}$$
$$+ f^r_{mn}\pi_{2r}\lambda^{2m}Q^{2na} + \frac{1}{3}f^m_{ne}f^e_{rq}\varepsilon_{cd}\pi_{2m}Q^{2rc}Q^{2nd}Q^{2qa}\Bigg), \qquad a = 1, 2. \tag{25}$$

The extended BRST invariant Hamiltonian, solution of equation (24), will be

$$H = H_c + \int \mathrm{d}^2x(P_{2ma}Q^{1ma} + \pi_{2m}\lambda^{1m}). \tag{26}$$

A great advantage of the extension of the BRST symmetry is given by the fact that the large set of variables which are introduced in a natural way can be used in the gauge-fixing procedure. We can construct many gauge-fixing functionals which lead to the gauge-fixing term. Particularly, for the $sp(2)$ BRST symmetry we can construct the following gauge-fixing functionals:

(i) $\overline{Y}$ with $\varepsilon(\overline{Y}) = 1$ and $gh(\overline{Y}) = -1$ st

$$H_{g\cdot f} = H + [\overline{Y}, \Omega^1 + \Omega^2],$$

(ii) $\{Y_a, a = 1, 2\}$ with $\varepsilon(Y_a) = 1, lev(Y_a) = 1 - a$ and $gh(Y_a) = -1$ st

$$H_{g\cdot f} = H + [Y_a, \Omega^a],$$

(iii) Y with $\varepsilon(Y) = 1, lev(Y) = -2$ and $gh(Y) = -2$ st

$$H_{g\cdot f} = H + \frac{1}{2!}\varepsilon_{ab}[\Omega^a, [\Omega^b, Y]].$$

All these gauge-fixing functionals lead to the same gauge-fixed Hamiltonian.

The concrete expressions of the previous functionals obtained by us for the Yang–Mills model show that Y, that is the last choice, has the most simple form: a product between a function of the original (real) variables and the last ghost momenta introduced in (19) for assuring the acyclicity of $\{\delta_a, a = 1, 2\}$.

So, the gauge-fixed Hamiltonian will be given by

$$H_Y = H + \frac{1}{2!}\varepsilon_{ab}[\Omega^a, [\Omega^b, Y]], \tag{27}$$

where the gauge-fixing fermion has the form

$$Y = \int \mathrm{d}^2x(\partial_i A^{im})\pi_{1m}. \tag{28}$$

The gauge-fixed action will be

$$S_Y = \int \mathrm{d}^3x\Bigg(-\frac{1}{4}F^{\mu\nu}_m F^m_{\mu\nu} + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\left(\partial_\mu A^m_\nu A^m_\rho + \frac{1}{3}f^m_{nr}A^m_\mu A^n_\nu A^r_\rho\right)$$
$$+ \mathrm{i}\overline{\psi}_{\alpha m}(\gamma^\mu)^m_n(D_\mu)^n_r\psi^{\alpha r} + (\partial_\mu P_{1ma})(D^\mu)^m_n Q^{2na} - (\partial_\mu\pi_{1m})(D^\mu)^m_n\lambda^{2n}\Bigg). \tag{29}$$

4. From Hamilton to Lagrange

There is a direct modality of obtaining the $sp(2)$ BRST Lagrangian formalism [7]. It assumes the use of a very large spectrum of ghost generators in order to obtain a gauge-fixed action, spectrum appearing because the gauge-fixing functional is chosen by using one of the two generalized antibrackets, in respect of which all the antifields have to have canonical partners. To avoid this unuseful extension we will construct the Lagrangian formalism following equivalence with the Hamiltonian one. By this, a simple gauge-fixing term of the form (28) can be transferred using some non-canonical variables. All these facts are presented by many authors, as for example in [17] for the standard theory and in [11] for the $sp(2)$ formalism. We will develop here the equivalence between the Hamiltonian and the Lagrangian formalism at the level of the $sp(2)$ approach, particularly applied for the model given by (4). We will start by implementing in this action the first class constraints (13) and (15) by means of the Lagrange multipliers. By doing so we will have the following extended action:

$$S_{can}\left[A^i_m, \psi^{\alpha m}, A^m_0, u^{1m}\right] = \int \mathrm{d}^3x \left(\frac{1}{2}p^m_i p^m_i + \frac{k}{4\pi}\varepsilon^{0ij} p^m_i A^m_j + \frac{1}{4}F^{ij}_m F^m_{ij} - \frac{1}{2}\left(\frac{k}{4\pi}\right)^2 A^i_m A^m_i - A^m_0\left((D_i)^n_m p^i_n + \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_{jm} + \mathrm{i}\overline{\psi}_{\alpha q}(\gamma^0)^q_n f^n_{rm}\psi^{\alpha r}\right) - \mathrm{i}\overline{\psi}_{\alpha m}(\gamma^j)^m_n (D_j)^n_r \psi^{\alpha r} + u^{1m} p^0_m\right). \tag{30}$$

It remains invariant at the gauge transformations

$$\delta_\varepsilon A^0_m = 2(D^0)^n_m \varepsilon_n, \qquad \delta_\varepsilon A^i_m = (D^i)^n_m \varepsilon_n, \qquad \delta_\varepsilon \psi^{\alpha m} = f^m_{nr}\psi^{\alpha r}\varepsilon^n, \tag{31}$$

where

$$\varepsilon_{2n} = \dot{\varepsilon}_n, \qquad \varepsilon_1 n = -(D^0)^m_n \varepsilon_m \tag{32}$$

represent the gauge parameters. In fact $\{\varepsilon_{1m}, \varepsilon_{2m}\}$ are adopted as the first set of ghosts $\{Q^{\Delta ma}, \Delta = 1, 2\}$ in the Lagrangian context. Because of (32), we note that a unique set of variables, let say Q^{2ma}, is independent and we can work with them alone. As the $sp(2)$ formalism asks for doubling the gauge transformations, the theory becomes reducible of the first order and demands for ghosts of ghosts, λ^{2m} [7]. By this, all the variables (22) which generate the Hamiltonian field sector are integrated in the Lagrangian extended space, too. In addition, the Lagrangian field spectrum will contain the Lagrange multipliers u^{1m} among the generators. Consequently, the $sp(2)$ Lagrangian field spectrum will be

$$\{q, u^{1m}, Q^A\} = \{A^{\mu m}, \psi^{\alpha m}, \overline{\psi}_{\alpha m}, u^{1m}, Q^{2ma}, \lambda^{2m}\}. \tag{33}$$

As in the $sp(2)$ Lagrangian case a unique BRST generator is defined and doubling is induced by two different antibrackets, two different sets of 'antifields', one for each antibracket, will ensure the canonical structure of the theory. They will be denoted by

$$\{q^*_a, u^*_{1ma}, Q^*_{Aa}\} = \{A^*_{\mu ma}, \psi^*_{\alpha ma}, \overline{\psi}^{*\alpha ma}, u^*_{1ma}, Q^*_{2mab}, \lambda^*_{2ma}, a = 1, 2\}. \tag{34}$$

In order to be able to make an equivalence between Hamiltonian and Lagrangian formalisms, we will adopt for the Lagrangian variables (33) and (34) the same graduation rules as for the Hamiltonian ones [10]. On this basis, we are able to do some identification among the antifields (34) and the ghost momenta (20):

$$u^*_{1ma} \equiv P_{1ma}, \qquad A^*_{0ma} \equiv P_{2ma}. \tag{35}$$

The previous identification does not completely solve the problem of equivalence between the Lagrangian and the Hamiltonian extended spaces. There are ghost momenta in (20) which are not yet transferred to the Lagrangian context. To do this, we require additional Lagrangian variables. We shall introduce the 'bar' variables [11]:

$$\{\overline{q}, \overline{u}_{1m}, \overline{Q}_A\} = \{\overline{A}_{\mu m}, \overline{\psi}_{\alpha m}, \overline{\overline{\psi}}^{\alpha m}, \overline{u}_{1m}, \overline{Q}_{2ma}, \overline{\lambda}_{2m}\}. \tag{36}$$

The whole correspondence between the generators of the extended spaces in the two formalisms will now be given by

- the same 'field' spectra (22);
- the Hamiltonian 'ghost momenta' sector will become in the Lagrangian context

$$P_{1ma} \equiv u^*_{1ma}, \qquad P_{2ma} \equiv A^*_{0ma}, \qquad \pi_{1m} \equiv \overline{u}_{1m}, \qquad \pi_{2m} \equiv \overline{A}_{0m}. \tag{37}$$

As we can remark, the extended Lagrangian spectrum is larger, including the Hamiltonian one. In the Lagrangian case each field $\{q, Q^A\}$ attaches two antifields $\{q^*_a, Q^*_{Aa}, a = 1, 2\}$ and the 'bar' antifields $\{\overline{q}, \overline{Q}_A\}$.

It is important to note that the 'bar' variables are unpaired variables. Their 'canonical' counterparts appear as superfluous in our designed model. The price we have to pay for this 'symmetry breaking' consists of imposing the Koszul–Tate differentials to contain a non-canonical part:

$$\delta_a * = \delta_a^{(c)} * + \delta_a^{(nc)} *, \qquad a = 1, 2. \tag{38}$$

Relation (38) will determine a similar decomposition for the BRST differentials

$$s_a * = s_a^{(c)} * + s_a^{(nc)} * \equiv (*, S)_a + V_a *, \qquad a = 1, 2. \tag{39}$$

The equivalence between the two formalisms, Hamiltonian and Lagrangian, will be expressed by imposing the requirement that the Lagrangian and Hamiltonian BRST operators have the same action on the common set of generators (22) and (35):

$$(*, S)_a + V_a * \equiv [*, \Omega_a], \qquad a = 1, 2. \tag{40}$$

The nilpotency condition for BRST differentials (39) leads to the master equations

$$\tfrac{1}{2}(S, S)_a + V_a S = 0, \qquad a = 1, 2. \tag{41}$$

The non-canonical operators $\{V_a, a = 1, 2\}$ will have the form

$$V_a * \equiv (-)^{\varepsilon(q)} \varepsilon_{ab} q^*_c \frac{\delta^R}{\delta \overline{q}} * + (-)^{\varepsilon(Q^A)+1} \delta_{ab} \overline{Q}_{Ab} \frac{\delta^R}{\delta \overline{Q}_A} *, \tag{42}$$

and are anticommuting

$$V_a V_b + V_b V_a = 0, \qquad a, b = 1, 2.$$

The $sp(2)$ BRST Lagrangian generator can be expressed as a sum of the form

$$S = S_1 + S_2 + \text{more}.$$

The decomposition has been done so that the first part S_1 is linear in the 'star' antifields attached to the real variables. It can be directly obtained from the gauge transformations (31). The second part S_2 will be processed from (40) and (25). The remaining part, *more*, will be given by requiring that S be the solution of the master equations (41).

We will pass now to the gauge-fixing problem and will propose a new procedure, based on the equivalence between the two BRST formalisms, the Hamiltonian and the Lagrangian ones. Our proposal takes into account exactly the existence of the non-canonical operators (42) and the equivalence requirement (40). It comprises:

the choice of the gauge-fixing functional starting from the Hamiltonian approach with clear gauge-fixing rules established in the third section;

all the antifields, except for the antifields attached to the Lagrange multipliers u^*_{1ma} and $\overline{u}_{1m}$, will be eliminated by using the non-canonical operators (42), on the basis of the following relations:

$$q^*_a = \frac{\delta}{\delta q^a}\left(\frac{1}{2}\varepsilon_{ab}V_bY\right), \qquad \overline{q} = \frac{\delta Y}{\delta q}$$
$$Q^*_{Aa} = \frac{\delta}{\delta Q^A}\left(\frac{1}{2}\varepsilon_{ab}V_bY\right), \qquad \overline{Q}_A = \frac{\delta Y}{\delta Q^A} \tag{43}$$

$$u^{1m} = -\frac{\delta}{\delta u^*_{1ma}}\left(\frac{1}{2}\varepsilon_{ab}V_bY\right) = -\frac{\delta Y}{\delta \overline{u}_{1m}}. \tag{44}$$

The concrete form of the gauge-fixing functional, taken, as mentioned, similarly to (28), will be

$$Y = \int \mathrm{d}^3x(\partial_\mu A^{\mu m})\overline{u}_{1m}. \tag{45}$$

The gauge-fixed solution of the master equations (41) will have the form

$$S' = S + \frac{1}{2!}\varepsilon^{ab}s_as_bY. \tag{46}$$

Now, we will eliminate some variables using relations (43) and (44). The only non-vanishing variables become

$$A^*_{\mu ma} = \frac{\delta}{\delta A^{\mu m}}\left(\frac{1}{2}\varepsilon_{ab}V_bY\right) = \partial_\mu P_{1ma},$$
$$\overline{A}_{\mu m} = \frac{\delta Y}{\delta A^{\mu m}} = \partial_\mu \pi_{1m},$$
$$u^{1m} = -\frac{\delta}{\delta u^*_{1ma}}\left(\frac{1}{2}\varepsilon_{ab}V_bY\right) = -\frac{\delta Y}{\delta \overline{u}_{1m}} = p^0_m.$$

Coming back with these relations to (46), the gauge-fixed action takes the form

$$S_Y[A,\psi,\overline{\psi},Q,P,\lambda,\pi] = \int \mathrm{d}^3x\Bigg(-\frac{1}{4}F^{\mu\nu}_m F^m_{\mu\nu} + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\left(\partial_\mu A^m_\nu A^m_\rho + \frac{1}{3}f^m_{nr}A^m_\mu A^n_\nu A^r_\rho\right)$$
$$+\mathrm{i}\overline{\psi}_{\alpha m}(\gamma^\mu)^m_n(D_\mu)^n_r\psi^{\alpha r} + (\partial_\mu P_{1ma})(D^\mu)^m_n Q^{2na} - (\partial_\mu\pi_{1m})(D^\mu)^m_n\lambda^{2n}\Bigg). \tag{47}$$

As we can see, it has a quite simple form which does not require additional antifields, as it happens with the gauge-fixing procedure usually applied to the pure extended BRST Lagrangian formalism [7].

5. Conclusions

The main focus of this paper was to find an extended BRST covariant formalism for a Chern–Simons model coupled with Dirac fields. It has been achieved starting from the equivalence between the Lagrangian and the Hamiltonian formulations. More than the expression of the extended action for this model, our research findings help to clarify what is the advantage of implementing the Lagrangian formalism not as a pure one but on the basis of its Hamiltonian counterpart.

The main difficulty in extending the standard BRST Lagrangian formalism toward an $sp(2)$ one lies in the unnatural and unnecessary enlargement of the ghost spectrum that the canonical structure preservation and the gauge fixing procedure would require. More precisely, in the pure $sp(2)$ Lagrangian approach, the usual gauge-fixing procedure supposes to maintain only one of the two antibrackets. By doing so, all the antifields canonical conjugated with the fields in the eliminated antibracket ask for conjugate variables in the maintained antibracket. So, even if we have already done the extension of the ghost spectrum involved by the $sp(2)$ formalism, a new extension of the generators has to be done at this stage. The way we followed avoid this extension. Moreover, the passage from the Hamiltonian to the Lagrangian formalism clearly explains the form of the $sp(2)$ master equation and the role of the non-canonical operators $\{V_a, a = 1, 2, 3\}$ in the gauge-fixing procedure. The choice of the gauge-fixing function of the form (45), induced by the equivalence between the two formalisms, is convenient but contains unpaired 'bar' variables. It asks for the use of the non-canonical operators in the elimination of the antifields, as in (43) and (44).

In conclusion, our approach based on the equivalence between the Lagrangian and the Hamiltonian formalism: *(i)* makes clear the role of the non-canonical operators $\{V_a, a = 1, 2, 3\}$ in limiting the antifields spectrum; *(ii)* leads to an optimized minimal sector, without unnecessary variables, and *(iii)* allows us to find a very simple gauge-fixing term. We have to mention that the non-canonical operators are already well known in the literature and that our contribution is related to obtaining their concrete form in a natural way. We suggested the use of these operators in the gauge-fixing procedure. In the traditional approach, the non-canonical operators do not have a clearly specified role. In contrast, in the equivalence-based approach, the operators $\{V_a, a = 1, 2, 3\}$ act as leading items, essentials in limiting the antifield spectrum and in finding a quite simple gauge-fixing term.

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